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Research of Gravitation in Flat Minkowski Space

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Abstract In this paper it is introduced and studied an alternative theory of gravitation in flat Minkowski space. Using an antisymmetric tensor ϕ, which is analogous to the tensor of electromagnetic field, a non-linear connection is introduced. It is very convenient for studying the perihelion/periastron shift, deflection of the light rays near the Sun and the frame dragging together with geodetic precession i.e. effects where angles are involved. Although the corresponding results are obtained in rather different way, they are the same as in the General Relativity. The results about the barycenter of two bodies are also the same as in the General Relativity. Comparing the derived equations of motion for the n-body problem with the Einstein-Infeld-Hoffmann equations, it is found that they differ from the EIH equations by Lorentz invariant terms of order c^{-2}.

Keywords Non-linear connection · Equations of motion · Lagrangian · n-body problem · Minkowski space

1 Introduction

In this paper, the gravitational phenomena are studied in flat Minkowski space and this approach is a small step ahead of the Special Relativity. In the literature there are some attempts the results of the General Relativity to be obtained in flat space-time and a study

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of such attempts and a proposed theory is given in [1]. Another example is the teleparallel approach [2], where the metric is hidden in the frame. Teleparallel gravity is reduced to General relativity and therefore calculations for the gravitational tests are not necessary. However, the study in this paper is broader and we also make the calculations (up to c^{-2}) to investigate the agreement with the basic gravitational tests.

For the equations of motion the position of the observer is also important, i.e. whether he is away from the gravitational field, or inside the gravitational field. Indeed, the equations depend only on the chosen coordinate system, but the parameters in the equations depend on the position of the observer in its local coordinate frame. Such position dependent parameters are for example the acceleration toward the gravitational bodies. So, we can distinguish four cases:

1. The observer is far from gravitation and the coordinates are orthonormal;
2. The observer is inside the gravitational field and the coordinates are ordinary (curvilinear);
3. The observer is inside the gravitational field and the coordinates are orthonormal;
4. The observer is far from gravitation and the coordinates are ordinary (curvilinear).

In this paper, we focus on the case 1. Specially, the theory will be covariant with respect to the Lorentz transformations with constant elements, analogously to the Special Relativity, because of the freedom of the choice of the inertial coordinate system far from gravitation, where the observer is placed. Cases 2 and 3 are much more complicated and they will be considered in a forthcoming paper. The case 4 is a subject of the General Relativity (GR), more precisely the Einstein-Infeld-Hoffmann equations, which will be supported in Sect. 7.5.

In Sect. 2 we present a nonlinear connection in the Minkowskian space. Such a research offers a great convenience in calculations which have been used as advantage also in some other approaches [3, 4], etc. It gives a very close relationship with the electrodynamics (Sect. 2) which gives possibility for quantization of gravitation and unification with the other interactions, since the other interactions are considered in flat space (see [5] for interesting discussions on the topic).

Non-linear connections are widely used at present time. For example, non-linear connections using Finsler geometry are studied in [6–8] and also in [9–13]. But although for studying gravitation both nonlinear connection and research in flat space are not new [14], in this paper we propose an approach obeying both characteristics.

We use *ict* convention (see pp. 51 in [15] about *ct*/*ict* conventions). So, we work with the Euclidean metric $\mathrm{diag}(1, 1, 1, 1)$ and upper and lower indices will not differ.

2 Introduction of a Non-linear Connection

Firstly, we explain why it is not convenient to use linear connection. Let us examine the effect of a linear connection concerning the 4-velocities. The parallel transport of any vector in the direction of a 4-vector of velocity (V_1, V_2, V_3, V_4) means that parallel transport is made in each of the four directions $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, these are multiplied by V_1, V_2, V_3, and V_4 respectively and then all is added together. But, we can not consider a 4-velocity as a translation, since that is not supported by the special-relativistic addition. Rather, a 4-velocity should be regarded as a Lorentz transformation with its incorporated hyperbolic properties in the 4-dimensional space. So, we will consider a non-linear connection in a sense that the condition $\nabla_{aX+bY} = a\nabla_X + b\nabla_Y$ is dismissed. The construction will be made in three steps.

2.1 Using an Analogy from Electromagnetism

Firstly, we will make a complete analogy with the electromagnetism, where instead of the charge e we will consider mass M, and instead of the potential $\frac{e}{r}$ we will consider the gravitational potential $\frac{GM}{r}$, assuming that M has the same value in each inertial coordinate system. Further we will introduce an antisymmetric tensor analogous to the tensor of electromagnetic field. We accept a priori that the velocity of the gravitational interaction is c, which would enable us to find this tensor when the source of gravitation field is accelerated.

Let us consider the motion of a test body under the influence of a gravitational body with mass M concentrated into a point with a time dependent 4-vector of velocity

$$(U_1, U_2, U_3, U_4) = \frac{1}{\sqrt{1-u^2/c^2}}\left(\frac{u_x}{ic}, \frac{u_y}{ic}, \frac{u_z}{ic}, 1\right), \tag{2.1}$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the corresponding 3-vector of velocity. Assume that the 4-vector of velocity of a test body with mass m is given by

$$(V_1, V_2, V_3, V_4) = \frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{v_x}{ic}, \frac{v_y}{ic}, \frac{v_z}{ic}, 1\right). \tag{2.2}$$

We build an antisymmetric tensor field ϕ_{ij}, in the following way. Firstly, we consider a special case when the sources of gravitation move with constant velocities. It is sufficient to define this tensor for a stationary body with point mass M. Then, using the Lorentz transformations and the principle of superposition of the fields, the tensor is theoretically well defined in this special case. In this case, at the point (x, y, z) ϕ is defined by

$$(\phi_{ij}) = \begin{bmatrix} 0 & 0 & 0 & \frac{GM}{r^3c^2}(x-x_0) \\ 0 & 0 & 0 & \frac{GM}{r^3c^2}(y-y_0) \\ 0 & 0 & 0 & \frac{GM}{r^3c^2}(z-z_0) \\ -\frac{GM}{r^3c^2}(x-x_0) & -\frac{GM}{r^3c^2}(y-y_0) & -\frac{GM}{r^3c^2}(z-z_0) & 0 \end{bmatrix}, \tag{2.3}$$

where (x_0, y_0, z_0) is the position of the gravitational body.

The 3-vector $c^2(\phi_{41}, \phi_{42}, \phi_{43})$ is the Newton acceleration toward the gravitational body, which is analogous to the electric field $\mathbf{E}$. The physical interpretation of the components ϕ_{ij} for $1 \le i, j \le 3$ will be given by (2.17).

Notice that using this tensor in flat Minkowski space it is obtained a general formula for frequency redshift/blueshift [16], which simultaneously explains the Doppler effect, gravitational redshift and under one cosmological assumption it also explains the cosmological redshift and the blueshift arising from the Pioneer anomaly. The gravitational redshift there is a consequence of the attraction force near the gravitational bodies and we do not need curved space any more.

Now let us consider arbitrary time variable vector $\mathbf{u}$ of the source of gravitation. Analogously as obtaining Lienard-Wiechert potentials in electrodynamics, the components of the tensor in case of gravitation can be obtained at each space-time point, using that the gravitational interaction transmits with velocity c. So, we get the following analogous formulae as in electrodynamics

$$c^2(\phi_{41}, \phi_{42}, \phi_{43}) = -\frac{GM}{(R - \frac{\mathbf{R}\cdot\mathbf{u}}{c})^3}\left(\mathbf{R} - \frac{\mathbf{u}}{c}R\right)$$

$$- \frac{GM}{c^2(R - \frac{\mathbf{R}\cdot\mathbf{u}}{c})^3}\mathbf{R} \times \left[\left(\mathbf{R} - \frac{\mathbf{u}}{c}R\right) \times \dot{\mathbf{u}}\right], \tag{2.4a}$$

$$\frac{c}{i}(\phi_{32}, \phi_{13}, \phi_{21}) = \frac{1}{R}\mathbf{R} \times (\phi_{41}, \phi_{42}, \phi_{43}). \tag{2.4b}$$

Here $\mathbf{u}$ is the velocity of the gravitational body, $\mathbf{R}$ is the 3-vector from the gravitational body to the considered point (x, y, z, ict) in the chosen coordinate system calculated at the space-time point (x', y', z', ict') of the gravitational body, such that after time $t - t'$ of transmission of the interaction, it arrives at the considered point (x, y, z, ict). Thus, t' appears as a solution of the equation

$$t = t' + \frac{R(t')}{c}. \tag{2.5}$$

In (2.4a) $\dot{\mathbf{u}} = \partial\mathbf{u}/\partial t'$ and $R = |\mathbf{R}|$.

In the special case when $\dot{\mathbf{u}} = 0$ (2.4a) reduces to

$$c^2(\phi_{41}, \phi_{42}, \phi_{43}) = -\frac{GM}{R^3}\mathbf{R}\frac{1 - \frac{u^2}{c^2}}{(1 - \frac{u^2}{c^2}\sin^2\theta)^{3/2}}, \tag{2.6}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{u}$, and $\mathbf{R}$ is the 3-vector from the gravitational body to the considered point at time t. This special case can be deduced directly from (2.3) using the Lorentz transformations.

In this paper we will work up to c^{-2} approximation. Since for the 2-body problem $\mathbf{R}$ is collinear with $\dot{\mathbf{u}}$, the last term in (2.4a) can be neglected for c^{-2} approximation. Hence, in this paper we can use the equality (2.6), except in Sect. 6, where the n-body problem is considered.

A natural question appears about the analog of the 4-vector potential from the electromagnetism. It is treated in some previous papers [17, 18] and it is not necessary to consider it in this paper.

We can resume, so far, that in case of gravitation we accepted some facts from the electromagnetism. But we must emphasize that there are two essential differences, which will be considered in the Sects. 2.2 and 2.3. The gravity is associated with a spin-2 field rather than the spin-1 field of the electromagnetism.

(i) While the charge e in electrodynamics is invariant scalar in all coordinate systems, the gravitational mass M is not invariant. Since the inertial mass is not Lorentz invariant according to the Special Relativity, it is naturally to expect that the gravitational mass is not invariant in flat Minkowski space. Thus, the tensor ϕ must be modified, and this will be made in Sect. 2.2.

(ii) The equations of motion can not simply copy the Lorentz formula from the electrodynamics, because it gives a parallel transportation only of a single vector, the 4-vector of velocity, but not of an arbitrary vector. Thus, in case of gravitation we must modify the Lorentz force, and it will be made in Sect. 2.3. Moreover, while the Lorentz force acting on the charged particles depends on the electromagnetic field at the considered point and not on the velocity of the source of the electromagnetic field, in case of gravitation, as we shall see, the motion depends on the source of gravitation very explicitly. This dependence in GR is implicitly contained in the Einstein's equations and it is explicitly visible in the Einstein-Infeld-Hoffmann equations.

2.2 Influence of the Masses to the Gravitational Force and Acceleration

A mass far from gravitation measured by an observer far from gravitation will be called proper mass and will be denoted by m, M, m_1, $m_2, \ldots$. An observer far from gravitation observing a body with proper mass m that has fallen into a gravitational field with gravitational potential $\frac{GM}{R}$, will measure $\frac{m}{1+\frac{GM}{Rc^2}}$ for the mass of the body. It is convenient the scalar $\mu = 1 + \frac{GM}{Rc^2}$ to call also (gravitational) potential. Assume that the test body has a small mass m with respect to the gravitational body. Then this is in accordance with the preserving of the energy in a gravitational field, such that considering also the kinetic energy, the mass $\frac{m}{1+\frac{GM}{Rc^2}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ will be unchanged up to c^{-2} during the motion of the test body.

Let us consider two bodies with masses m_1 and m_2 on a distance R between their centers. Then the mass m_1 is observed to be $\frac{m_1}{1+\frac{Gm_2}{Rc^2}}$ under the influence of the other mass m_2, and the mass m_2 is observed to be $\frac{m_2}{1+\frac{Gm_1}{Rc^2}}$ under the influence of the other mass m_1. So, the gravitational force which acts on the body with mass m_1 and is caused by the body with mass m_2 is given by

$$\mathbf{f} = \frac{m_1}{1 + \frac{Gm_2}{Rc^2}} \nabla \frac{Gm_2}{R(1 + \frac{Gm_1}{Rc^2})}, \tag{2.7}$$

while the acceleration of the body with mass m_1 is assumed to be

$$\mathbf{a} = \frac{1}{1 + \frac{Gm_2}{Rc^2}} \nabla \frac{Gm_2}{R(1 + \frac{Gm_1}{Rc^2})}. \tag{2.8}$$

The formulae (2.7) and (2.8) will be generalized below by (2.10) and (2.11). We must emphasize that these formulae are given with respect to an observer far from gravitation, assuming also that the observer does not move with respect to the gravitational bodies. Here, the distance R is a function of 6 coordinates: 3 coordinates of the body with mass m_1 and 3 coordinates of the body with mass m_2, and the gradient is taken with respect to the coordinates of the body with mass m_1. It is easy to see that up to c^{-2} the acceleration (2.8) can be written in the form

$$\mathbf{a} = -\frac{\mathbf{R}}{R} \frac{Gm_2}{R^2} \left(1 - \frac{G(2m_1 + m_2)}{Rc^2}\right), \tag{2.9}$$

where $\mathbf{R}$ is the vector from the body with mass m_2 towards the body with mass m_1.

In Sect. 6 we will consider the general equations for n-body problem. Then it will be necessary to use a more general formula for the acceleration. If we consider the interaction of two bodies, for example with masses m_1 and m_2, we must use that their masses in the gravitational field are

$$m_1 \Big/ \left[\left(1 + \frac{Gm_2}{r_{12}c^2}\right)\left(1 + \frac{Gm_3}{r_{13}c^2}\right) \cdots\right] \quad \text{and} \quad m_2 \Big/ \left[\left(1 + \frac{Gm_1}{r_{21}c^2}\right)\left(1 + \frac{Gm_3}{r_{23}c^2}\right) \cdots\right]$$

respectively, where r_{ij} is the distance between the bodies with masses m_i and m_j. Now, analogously to (2.7), and (2.8) for the force/acceleration of the body with mass m_1 caused by the mass m_2 we accept *axiomatically* that

$$\mathbf{f} = \frac{m_1}{(1 + \frac{Gm_2}{r_{12}c^2})(1 + \frac{Gm_3}{r_{13}c^2}) \cdots} \nabla \frac{Gm_2}{r_{12}(1 + \frac{Gm_1}{r_{12}c^2})(1 + \frac{Gm_3}{r_{23}c^2}) \cdots}, \tag{2.10}$$

$$\mathbf{a} = \frac{1}{(1+\frac{Gm_2}{r_{12}c^2})(1+\frac{Gm_3}{r_{13}c^2})\cdots}\nabla\frac{Gm_2}{r_{12}(1+\frac{Gm_1}{r_{12}c^2})(1+\frac{Gm_3}{r_{23}c^2})\cdots}. \tag{2.11}$$

Analogously to (2.9), in this general case we obtain

$$\mathbf{a} = \left[\left(1+\frac{Gm_2}{r_{12}c^2}\right)\left(1+\frac{Gm_1}{r_{12}c^2}\right)^2\left(1+\frac{Gm_3}{r_{13}c^2}\right)\right.$$
$$\left.\times\left(1+\frac{Gm_3}{r_{23}c^2}\right)\left(1+\frac{Gm_4}{r_{14}c^2}\right)\left(1+\frac{Gm_4}{r_{24}c^2}\right)\cdots\right]^{-1}\nabla\frac{Gm_2}{r_{12}}. \tag{2.12}$$

Notice that according to the assumptions that the observer is far from gravitation and the gravitational bodies do not move with respect to the observer, the acceleration from Sect. 2.1 is given by $\nabla\frac{Gm_2}{r_{12}}$. Thus, for moving bodies with respect to the observer, up to c^{-2}, the components $\phi_{14}, \phi_{24}, \phi_{34}, \phi_{41}, \phi_{42}, \phi_{43}$ should be multiplied by the coefficient in front of $\nabla\frac{Gm_2}{r_{12}}$ in (2.12). Since the components w_x, w_y, w_z are much smaller than a_x, a_y, a_z, we can conclude that *all the components of the tensor ϕ should be multiplied by the coefficient in front of $\nabla\frac{Gm_2}{r_{12}}$ in* (2.12). This coefficient in (2.12) is a scalar in the Minkowskian space up to c^{-2}, and hence the product, i.e. the modified tensor ϕ, will preserve its tensor character. We agree that further on, ϕ will always mean this modified tensor.

We shall draw some conclusions. For example, if m_1 is negligible small mass and $m_2 = M$ is non-zero mass of a stationary body, then the acceleration of the body with mass m_1 is equal to $\mathbf{a} = -\frac{\mathbf{R}}{R}\frac{\frac{GM}{R^2}}{1+\frac{GM}{Rc^2}}$. This acceleration can be written as

$$\mathbf{a} = c^2\nabla\ln\left(1+\frac{GM}{Rc^2}\right). \tag{2.13}$$

Now, it is clear that the potentials $\mu = 1 + \frac{GM}{Rc^2}$ and $C\mu$, where C is a constant, lead to the same acceleration.

At the end of this subsection we emphasize the following remark about the preserving the energy of a system of n-bodies with masses $m_1, m_2, \ldots, m_n$. According to the accepted change of the mass near gravitational bodies, the energy of the i-th body, including the energy in rest $m_i c^2$ is equal to

$$\frac{m_i c^2}{\sqrt{1-\frac{v_i^2}{c^2}}\prod\limits_{j\neq i}(1+\frac{Gm_j}{r_{ij}c^2})}.$$

Following the electrodynamics analogy, as the density of energy caused by charged particles is given by $\frac{E^2+H^2}{8\pi}$, in case of gravitation we have that the density of energy is given by $\frac{\mathbf{a}^2+\mathbf{w}^2c^2}{8\pi G}$, because $c\mathbf{w}$ corresponds to the magnetic field (see (2.17)). Since $w \sim c^{-2}$, this energy density can be replaced by $\frac{\mathbf{a}^2}{8\pi G}$. Hence the total energy is given by

$$\sum_{i=1}^{n}\frac{m_i c^2}{\sqrt{1-\frac{v_i^2}{c^2}}\prod_{j\neq i}(1+\frac{Gm_j}{r_{ij}c^2})} + \frac{1}{8\pi G}\int \mathbf{a}^2 dV. \tag{2.14}$$

Using that $\frac{1}{8\pi G}\int \mathbf{a}^2 dV = \sum_{i,j,j\neq i}\frac{Gm_i m_j}{2r_{ij}} + \text{const.}$, we obtain that up to a constant summand the total energy can be written in the form

$$\sum_{i=1}^{n}\frac{m_i c^2}{\sqrt{1-\frac{v_i^2}{c^2}}} - \frac{1}{8\pi G}\int \mathbf{a}^2 dV.$$

This formula is the same as in GR, and hence the conclusion in GR that the density of energy is $-\frac{\mathbf{a}^2}{8\pi G}$ [19], instead of $\frac{\mathbf{a}^2}{8\pi G}$. Although this energy $\int \frac{\mathbf{a}^2}{8\pi G} dV$ and also the kinetic energy take part in determining the barycenter of the system of bodies, both energies do not contribute to the acceleration of the other bodies. Only the mass $m_i/\prod_{j\neq i}(1+\frac{Gm_j}{r_{ij}c^2})$ plays role in the acceleration towards the i-th body. This is visible from (2.10) and (2.11). Also notice that the barycenter of the bodies remains unchanged, compared with the GR. Namely, analogously as obtaining the barycenter in the GR and in electrodynamics [19], in this case one obtains again the same radius-vector

$$\mathbf{r}_b = \frac{\sum_{i=1}^{n}\mathbf{r}_i(m_i c^2 + \frac{1}{2}m_i v_i^2 - \frac{Gm_i}{2}\sum_{j\neq i}\frac{m_j}{r_{ij}})}{\sum_{i=1}^{n}(m_i c^2 + \frac{1}{2}m_i v_i^2 - \frac{Gm_i}{2}\sum_{j\neq i}\frac{m_j}{r_{ij}})}. \tag{2.15}$$

2.3 Equations of Parallel Displacement

Notice that in a system of four orthonormal vectors A_{i1}, A_{i2}, A_{i3} and A_{i4}, where $A_{i\alpha}$ is the i-th coordinate of the α-th vector, using that $A_{i\alpha}$ is an orthogonal matrix, i.e. $AA^T = I$, the following tensor

$$\frac{dA_{i\alpha}}{ds}A_{j\alpha}, \tag{2.16}$$

$ds = ic\sqrt{1-\frac{v^2}{c^2}}dt$ is also skew-symmetric as ϕ_{ij} is. The formula (2.16) is invariant under the linear transformation $A_{i\alpha}\to B_{i\alpha} = A_{i\beta}R_{\beta\alpha}$, where R is an orthogonal matrix with constant elements. In the special case when $U_i = V_i$, we assume that the two tensors, ϕ_{ij} and the tensor in (2.16), are equal. Then the physical interpretation of ϕ_{ij} can be obtained using the tensor (2.16). Since (2.16) is invariant under the linear transformation $A\to AR$, without loss of generality, we may assume that $A_{ij} = \delta_{ij}$ at the considered point, and hence the components of (2.16) are 3-vector of acceleration and 3-vector of angular velocity. We represent ϕ in the following form

$$\phi = \begin{bmatrix} 0 & -i\omega_z/c & i\omega_y/c & -a_x/c^2 \\ i\omega_z/c & 0 & -i\omega_x/c & -a_y/c^2 \\ -i\omega_y/c & i\omega_x/c & 0 & -a_z/c^2 \\ a_x/c^2 & a_y/c^2 & a_z/c^2 & 0 \end{bmatrix}, \tag{2.17}$$

where $\mathbf{a} = (a_x, a_y, a_z)$ is the 3-vector of acceleration and $\mathbf{w} = (w_x, w_y, w_z)$ is the 3-vector of angular velocity. Indeed we accept the following notations

$$a_x = \phi_{41}c^2 = -\phi_{14}c^2, \qquad a_y = \phi_{42}c^2 = -\phi_{24}c^2, \qquad a_z = \phi_{43}c^2 = -\phi_{34}c^2,$$
$$w_x = ic\phi_{23} = -ic\phi_{32}, \qquad w_y = ic\phi_{31} = -ic\phi_{13}, \qquad w_z = ic\phi_{12} = -ic\phi_{21}.$$

In the special case when $(U_i) = (0, 0, 0, 1)$, from (2.3) it follows that $\mathbf{w} = (0, 0, 0)$. If $(U_i) \neq (0, 0, 0, 1)$, then $\mathbf{w}$ can be nonzero, analogously as for frame dragging.

Now, let us consider the general formula for the parallel transport of the considered frame $A_{i\alpha}$ in direction of the 4-vector of velocity V_i. We introduce the tensor $P = P(U, V)$ given by

$$P_{ij} = \delta_{ij} - \frac{1}{1 + U_s V_s}(V_i V_j + V_i U_j + U_i V_j + U_i U_j) + 2U_j V_i, \tag{2.18}$$

and accept *axiomatically* the following relationship between the tensor ϕ_{ij} and the tensor given by (2.16)

$$\frac{dA_{i\alpha}}{ds} A_{j\alpha} = P_{ri}\phi_{rk}P_{kj}, \tag{2.19}$$

or in matrix form $\frac{dA}{ds}A^T = P^T\phi P$. Notice that both sides of (2.19) are skew-symmetric matrices.

The tensor P_{ij} is an orthogonal matrix. It can be verified by using the identities $U_i U_i = V_i V_i = 1$. Moreover, it has the following property $P(U, V) = P(V, U)^{-1}$. Some other properties of this tensor are given in [20] and a justification for its appearance in (2.19) is given in [21]. For example, it is shown that using the standard addition, one can not uniquely determine a 4-vector in the Minkowskian space-time which would represent a relative 4-velocity of a point B with respect to a point A, assuming that B moves with 4-velocity V and A moves with 4-velocity U. So, the tensor $P(U, V)$ provides a transition between velocities, i.e. $P_{ij}U_j = V_j$. The tensor P with some of its properties was independently found also by other authors [22, 23].

In the special case $(U_i) = (0, 0, 0, 1)$, the tensor $P(U, V)$ is given by

$$P = \begin{bmatrix} 1 - \frac{1}{\nu}V_1^2 & -\frac{1}{\nu}V_1V_2 & -\frac{1}{\nu}V_1V_3 & V_1 \\ -\frac{1}{\nu}V_2V_1 & 1 - \frac{1}{\nu}V_2^2 & -\frac{1}{\nu}V_2V_3 & V_2 \\ -\frac{1}{\nu}V_3V_1 & -\frac{1}{\nu}V_3V_2 & 1 - \frac{1}{\nu}V_3^2 & V_3 \\ -V_1 & -V_2 & -V_3 & V_4 \end{bmatrix}, \tag{2.20}$$

where V_1, V_2, V_3, V_4 are given by (2.2), $\nu = 1 + V_4$, and this represents just a Lorentz transformation (as a boost, without space rotation). Multiplying (2.19) by $A_{j\beta}$ and sum for j we get

$$\frac{dA_{i\beta}}{ds} = P_{ri}\phi_{rk}P_{kj}A_{j\beta}, \tag{2.21}$$

and hence for the parallel displacement of an arbitrary vector A_i we get

$$\frac{dA_i}{ds} = P_{ri}\phi_{rk}P_{kj}A_j. \tag{2.22}$$

Particularly, for $A_i = V_i$, we obtain the equations of motion

$$\frac{dV_i}{ds} = P_{ri}\phi_{rk}P_{kj}V_j. \tag{2.23}$$

The last equation ($i = 4$) of (2.23) is a consequence of the first three equations, because if we multiply (2.23) by V_i and sum for $i = 1, 2, 3, 4$ we obtain the identity $0 = 0$. The same is true for (2.22) also.

Notice that the vectors U_i and V_i are tangent vectors of different curves, parameterized for example via the time parameters. Thus in (2.23) and the previous formulae, the 4-vector U_i should be taken at the point (x', y', z', ict'), where t and t' are related by (2.5), because we must take into account the time which is needed for the gravitational interaction to reach the test body. But, if we take the values of U_i at the same time t as the 4-vector V_i, then the acceleration of the test body would be changed of order c^{-4}, and so we will do that in this paper.

In the special case when $(U_i) = (0, 0, 0, 1)$, the nonlinear connection given by (2.22) and (2.23) is approximated [17] by a linear but not metric connection, using Christoffel symbols Γ^i_{jk}, such that $\Gamma^i_{jk} = -\Gamma^j_{ik}$. The Christoffel symbols depend on the components of the tensor ϕ and it is verified that the Einstein equations are satisfied up to c^{-2} for such a connection.

Two characteristics are essential for these equations of motion in flat space: *They are Lorentz invariant and they do not use any special coordinate system*. But the inertial and gravitational masses are different.

Let us consider the case of only one gravitational body in rest. Then the tensor ϕ_{ij} and the equations of motion are invariant under the transformation $\mu \to C\mu$, where C is a constant. Thus the tensor ϕ and the equations of motion are invariant under the gauge transformation $\ln\mu \to C + \ln\mu$, which is analogous to the Newtonian gauge transformation $V \to V + C$, and analogous to the invariance of the equations of motion in metric theories with respect to the transformation $g_{ij} \to C \cdot g_{ij}$.

3 Geodesics Applied to Planetary Orbits, Light Ray Trajectories and Gyroscope Precession

Our coordinate origin will be chosen to be at the center of the Sun, $U_1 = U_2 = U_3 = 0$ and the mass of each planet is assumed to be negligible with respect to the mass of the Sun.

A straight calculation of the matrix $S = P^T \phi P$, where ϕ is given by (2.17) and P is given by (2.20), leads to

$$
\begin{aligned}
S_{41} &= -S_{14} = i\frac{\omega_z}{c}V_2 - i\frac{\omega_y}{c}V_3 + \frac{a_x}{c^2}\left(V_4 + \frac{(V_1)^2}{1+V_4}\right) + \frac{a_y}{c^2}\frac{V_1V_2}{1+V_4} + \frac{a_z}{c^2}\frac{V_1V_3}{1+V_4},\\
S_{42} &= -S_{24} = i\frac{\omega_x}{c}V_3 - i\frac{\omega_z}{c}V_1 + \frac{a_x}{c^2}\frac{V_1V_2}{1+V_4} + \frac{a_y}{c^2}\left(V_4 + \frac{(V_2)^2}{1+V_4}\right) + \frac{a_z}{c^2}\frac{V_2V_3}{1+V_4},\\
S_{43} &= -S_{34} = i\frac{\omega_y}{c}V_1 - i\frac{\omega_x}{c}V_2 + \frac{a_x}{c^2}\frac{V_1V_3}{1+V_4} + \frac{a_y}{c^2}\frac{V_2V_3}{1+V_4} + \frac{a_z}{c^2}\left(V_4 + \frac{(V_3)^2}{1+V_4}\right),\\
S_{32} &= -S_{23} = \frac{a_z}{c^2}V_2 - \frac{a_y}{c^2}V_3 + i\frac{\omega_x}{c}\left(V_4 + \frac{(V_1)^2}{1+V_4}\right) + i\frac{\omega_y}{c}\frac{V_1V_2}{1+V_4} + i\frac{\omega_z}{c}\frac{V_1V_3}{1+V_4},\\
S_{13} &= -S_{31} = \frac{a_x}{c^2}V_3 - \frac{a_z}{c^2}V_1 + i\frac{\omega_x}{c}\frac{V_1V_2}{1+V_4} + i\frac{\omega_y}{c}\left(V_4 + \frac{(V_2)^2}{1+V_4}\right) + i\frac{\omega_z}{c}\frac{V_2V_3}{1+V_4},\\
S_{21} &= -S_{12} = \frac{a_y}{c^2}V_1 - \frac{a_x}{c^2}V_2 + i\frac{\omega_x}{c}\frac{V_1V_3}{1+V_4} + i\frac{\omega_y}{c}\frac{V_2V_3}{1+V_4} + i\frac{\omega_z}{c}\left(V_4 + \frac{(V_3)^2}{1+V_4}\right),\\
S_{11} &= S_{22} = S_{33} = S_{44} = 0.
\end{aligned}
\tag{3.1}
$$

Now by using equalities (3.1), (2.23) become

$$\frac{dv_x}{dt} = \left[(2-\beta^{-2})a_x - \frac{v_x}{c^2}(a_i v_i)\cdot\left(2+\frac{1}{\beta(\beta+1)}\right) + 2(v_y w_z - v_z w_y)\right], \tag{3.2a}$$

$$\frac{dv_y}{dt} = \left[(2-\beta^{-2})a_y - \frac{v_y}{c^2}(a_i v_i)\cdot\left(2+\frac{1}{\beta(\beta+1)}\right) + 2(v_z w_x - v_x w_z)\right], \tag{3.2b}$$

$$\frac{dv_z}{dt} = \left[(2-\beta^{-2})a_z - \frac{v_z}{c^2}(a_i v_i)\cdot\left(2+\frac{1}{\beta(\beta+1)}\right) + 2(v_x w_y - v_y w_x)\right], \tag{3.2c}$$

$$\frac{d}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{c^2}(a_i v_i), \tag{3.2d}$$

where $\beta = (1-\frac{v^2}{c^2})^{-1/2}$ and $a_i v_i = a_x v_x + a_y v_y + a_z v_z$. Indeed, (3.2d) is a direct consequence of (2.23) for $i=4$, and then this equality multiplied by $-\frac{v_x}{ic}$, $-\frac{v_y}{ic}$, and $-\frac{v_z}{ic}$ should be added to (2.23) for $i=1,2,3$, respectively in order to find dv_x/ds, dv_y/ds, and dv_z/ds. It is easy to verify that *if* $U=(0,0,0,1)$*, then* (3.2d) *does not depend on the matrix transformation* P *applied to* ϕ*, i.e.* (3.2d) *remains unchanged if we take* ϕ *instead of* $P^T\phi P$ *in* (2.23).

We will apply these equations in our special case. Using that $\mu = 1+\frac{GM}{rc^2}$, where M is the mass of the Sun, from (2.3) and (2.13) we obtain

$$(\phi_{ij}) = \begin{bmatrix} 0 & 0 & 0 & \frac{GM}{\mu r^3 c^2}x \\ 0 & 0 & 0 & \frac{GM}{\mu r^3 c^2}y \\ 0 & 0 & 0 & \frac{GM}{\mu r^3 c^2}z \\ -\frac{GM}{\mu r^3 c^2}x & -\frac{GM}{\mu r^3 c^2}y & -\frac{GM}{\mu r^3 c^2}z & 0 \end{bmatrix}, \tag{3.3}$$

and from the equations of motion (2.23), where the vector V_i is given by (2.2), can be found the components $\frac{d^2x}{dt^2} = \frac{dv_x}{dt}$, $\frac{d^2y}{dt^2} = \frac{dv_y}{dt}$, and $\frac{d^2z}{dt^2} = \frac{dv_z}{dt}$. We replace $v_z = 0$, assuming that the test body moves in the xy-plane, and thus, the equation for $i=3$ will be omitted. In this case, without any approximation, (3.2a), (3.2b), and (3.2d) reduce to

$$\frac{d^2x}{dt^2} = \frac{GM}{\mu r^3}\left[(\beta^{-2}-2)x + \frac{v_x}{c^2}(xv_x+yv_y)\left(2+\frac{1}{\beta(\beta+1)}\right)\right], \tag{3.4a}$$

$$\frac{d^2y}{dt^2} = \frac{GM}{\mu r^3}\left[(\beta^{-2}-2)y + \frac{v_y}{c^2}(xv_x+yv_y)\left(2+\frac{1}{\beta(\beta+1)}\right)\right], \tag{3.4b}$$

$$\beta - \ln\left(1+\frac{GM}{rc^2}\right) = \text{const.}, \tag{3.4c}$$

where (3.4c) is a solution of the differential equation (3.2d). This equation can be written in the following form

$$U_i V_i - \ln\left(1+\frac{GM}{rc^2}\right) = \text{const.}, \tag{3.4d}$$

where U_i is the 4-vector of velocity of the Sun and V_i is the 4-vector of velocity of the considered planet neglecting its mass. The scalars $U_i V_i$ and the 3-dimensional distance r determined in the system where the Sun rests, are invariant of the choice of the inertial coordinate system. Thus, (3.4d) is Lorentz invariant scalar equation. Indeed, the left side of (3.4d) is proportional with the Hamiltonian, or more precisely the Hamiltonian is given by

$$\mathcal{H} = mc^2\left(U_i V_i - \ln\left(1 + \frac{GM}{rc^2}\right)\right) \tag{3.4e}$$

where the mass m of the test body is negligible with respect to the gravitational mass M. According to the previous discussion, it does not depend on the matrix transformation P. *Thus, P does not influence the energy of the moving body, but it influences only the angular momentum of the moving body*. The previous discussion will continue in Sect. 7.4, where the Lagrangian will be given.

Using that

$$\frac{d\varphi}{dt} = \frac{d}{dt}\arctan\frac{y}{x} = \frac{v_y x - v_x y}{r^2}$$

for any angle φ, from (3.4a) and (3.4b), we obtain

$$\begin{aligned}\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) &= \frac{d^2y}{dt^2}x - \frac{d^2x}{dt^2}y = \frac{GM}{\mu r^3c^2}(v_y x - v_x y)(xv_x + yv_y)\left(2 + \frac{1}{\beta(\beta+1)}\right)\\ &= -r^2\frac{d\varphi}{dt}\frac{GM}{\mu c^2}\left(2 + \frac{1}{\beta(\beta+1)}\right)\frac{d}{dt}\left(\frac{1}{r}\right).\end{aligned}$$

Two cases will be considered.

3.1 Perihelion Shift

Assume that $v \ll c$, and consider the planetary orbits. Then $2 + \frac{1}{\beta(\beta+1)} \approx 2.5$ so neglecting the expressions of order c^{-4} we can switch to

$$\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) = -\frac{5}{2}\frac{GM}{c^2}\left(r^2\frac{d\varphi}{dt}\right)\frac{d}{dt}\left(\frac{1}{r}\right).$$

The solution of the previous equation is

$$r^2\frac{d\varphi}{dt} = C_2\exp\left(\frac{-5}{2}\frac{GM}{rc^2}\right), \quad C_2 = \text{const.} \tag{3.5}$$

Further, using the metric $(dr)^2 + r^2(d\varphi)^2 - c^2t^2 = ds^2$ in the flat space of Minkowski, we obtain

$$\left(\frac{dr}{dt}\right)^2 + \left(\frac{rd\varphi}{dt}\right)^2 = v^2, \qquad \left(r^{-2}\frac{dr}{d\varphi}\right)^2 + r^{-2} = v^2\left(r^2\frac{d\varphi}{dt}\right)^{-2},$$

and $\rho = r^{-1}$ satisfies the equation

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = v^2C_2^{-2}\exp\left(\frac{5GM\rho}{c^2}\right). \tag{3.6}$$

We are going to find v^2 from (3.4a) and (3.4b). By adding (3.4a) multiplied by $2v_x = 2dx/dt$ and (3.4b) multiplied by $2v_y = 2dy/dt$ and using that $xv_x + yv_y = rdr/dt$, we get

$$\begin{aligned}\frac{dv^2}{dt} &= \frac{GM}{\mu r^3}\left[(\beta^{-2}-2)2(xv_x+yv_y)+2\frac{v^2}{c^2}(xv_x+yv_y)\cdot\frac{5}{2}\right]\\ &= \frac{GM}{\mu r^2}\frac{dr}{dt}\left(-2+3\frac{v^2}{c^2}\right)=\frac{2GM}{\mu}\frac{d\rho}{dt}\left(1-\frac{3}{2}\frac{v^2}{c^2}\right).\end{aligned}$$

So, we obtain the following differential equation

$$\left(1-\frac{3}{2}\frac{v^2}{c^2}\right)^{-1}\frac{dv^2}{dt}=\frac{2GM}{\mu}\frac{d\rho}{dt}.$$

Replacing $1/\mu$ with $1-\frac{GM\rho}{c^2}$ in the previous differential equation and after some transformations, it becomes

$$\frac{-2}{3}c^2\frac{d\ln(1-\frac{3}{2}\frac{v^2}{c^2})}{dt}=2GM\frac{d}{dt}\left(\rho-\frac{GM\rho^2}{2c^2}+C\right).$$

The solution by v^2 is given by

$$v^2=2GM\left(\rho-\frac{GM\rho^2}{2c^2}+C\right)-3\frac{G^2M^2(\rho+C)^2}{c^2}.$$

After replacing this value in (3.6) we obtain

$$\left(\frac{d\rho}{d\varphi}\right)^2+\rho^2=A+B\rho+\frac{6G^2M^2}{c^2C_2^2}\rho^2. \tag{3.7}$$

Using that $C_2=\sqrt{GMa(1-\epsilon^2)}$, where a is the semi-major axis and ϵ is the eccentricity, standard calculations for the perihelion shift per orbit leads to the known result

$$\Delta\varphi=\frac{6GM\pi}{ac^2(1-\epsilon^2)}. \tag{3.8}$$

3.2 Deflection of the Light Rays Near the Sun

Let us consider the trajectory of a light ray near the Sun. We denote by R the radius of the Sun. In this case $\beta\to\infty$, so

$$\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right)=-2\frac{GM}{c^2}\left(r^2\frac{d\varphi}{dt}\right)\frac{d}{dt}\left(\frac{1}{r}\right)$$

and its solution is

$$r^2\frac{d\varphi}{dt}=C_2\exp\left(-2\frac{GM}{rc^2}\right),\qquad C_2=\text{const}. \tag{3.9}$$

Analogously to (3.6) we obtain

$$\left(\frac{d\rho}{d\varphi}\right)^2+\rho^2=v^2C_2^{-2}\exp\left(\frac{4GM\rho}{c^2}\right)$$

and replacing $v = c$, we get

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = c^2 C_2^{-2} \exp\left(\frac{4GM\rho}{c^2}\right). \tag{3.10}$$

The last step was possible because it is easy to verify that the light has a constant velocity c in a gravitational field in orthonormal coordinates.

If $r = R$, then $R\frac{d\varphi}{dt} = c$ and from (3.9) we get $C_2 = Rc\exp(\frac{2GM}{Rc^2})$. By replacing this value of C_2 into (3.10) we get

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = \frac{1}{R^2}\exp\left(\frac{4GM}{c^2}\left(\rho - \frac{1}{R}\right)\right),$$

$$\left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 = \frac{1}{R^2} - \frac{4GM}{R^3c^2} + \frac{4GM}{R^2c^2}\rho. \tag{3.11}$$

From (3.11) φ can be determined as a function of ρ:

$$\varphi = \arccos\frac{\rho R^2c^2 - 2GM}{Rc^2 - 2GM}, \tag{3.12}$$

such that $\varphi = 0$ if $\rho = \frac{1}{R}$. It is easy to conclude from (3.12) that the angle of deflection of a light ray near the Sun is equal to $\frac{4GM}{Rc^2}$. In [17] is given a different proof for this angle.

3.3 Geodetic Precession and the Frame Dragging Effect

Now we will deduce the formula for geodetic precession, simplifying that the gravitational body rests in the chosen coordinate system and hence also $\mathbf{w} = (0, 0, 0)$. We parallel transport the frame $A_{i\alpha}$ from Sect. 2.3, and assume that at the initial moment it is given by the matrix (2.20). Then we calculate the components $S_{3j}A_{j2} - S_{2j}A_{j3}$, $S_{1j}A_{j3} - S_{3j}A_{j1}$, $S_{2j}A_{j1} - S_{1j}A_{j2}$, where the matrix S is the same matrix given by (3.1). Straight calculation of these components yields

$$S_{3j}A_{j2} - S_{2j}A_{j3} = 3i\frac{a_yv_z - a_zv_y}{c^3}, \qquad S_{1j}A_{j3} - S_{3j}A_{j1} = 3i\frac{a_zv_x - a_xv_z}{c^3},$$
$$S_{2j}A_{j1} - S_{1j}A_{j2} = 3i\frac{a_xv_y - a_yv_x}{c^3}.$$

So, according to (2.21) we find

$$\frac{d(A_{32} - A_{23})/2}{ds} = 3i\frac{a_yv_z - a_zv_y}{2c^3}, \qquad \frac{d(A_{13} - A_{31})/2}{ds} = 3i\frac{a_zv_x - a_xv_z}{2c^3},$$
$$\frac{d(A_{21} - A_{12})/2}{ds} = 3i\frac{a_xv_y - a_yv_x}{2c^3}. \tag{3.13}$$

Having the transported matrix A, the following vector $\frac{1}{2}(A_{32} - A_{23}, A_{13} - A_{31}, A_{21} - A_{12})$, represents just the 3-vector of the small space rotation. Hence for the required angular velocity we obtain the known GR formula

$$\mathbf{\Omega} = \frac{3}{2}\frac{\mathbf{v} \times \mathbf{a}}{c^2}, \tag{3.14}$$

which is confirmed to about 0.7% using Lunar laser ranging data [24, 25], and the recent GPB experiment.

In the previous phenomena the source of gravitation was in rest, but for frame dragging effect it is necessary to consider a moving source. In the system where the source rests the tensor ϕ is well known and hence it is well known in any other system. Similar calculations to the previous yield the same formula for frame dragging [26] as the GR formula [27].

4 Periastron Shift of the Binary Systems

Let us consider an arbitrary binary system, for example a pulsar and its companion. In this section the periastron shift of the binary system will be calculated, assuming that both bodies are moving in the xy-plane. Let m be the mass of a pulsar and M be the mass of its companion, and let us choose the coordinate system such that at the initial moment $\mathbf{r}_b=(0,0,0)$ and $\frac{\mathbf{r}_b}{dt}=(0,0,0)$, where $\mathbf{r}_b$ is the barycenter (2.15) of the two bodies. We denote by $(x,y,0)$ the coordinates of the pulsar, and by $(x',y',0)$ the coordinates of its companion.

Let the 4-vectors of velocity of the pulsar and its companion are given by (2.2) and (2.1) respectively, where $u_z=v_z=0$. It is convenient to use the notations

$$R=\sqrt{(x-x')^2+(y-y')^2},\qquad r=\sqrt{x^2+y^2},\qquad \rho=1/r,\qquad r\approx\frac{M}{M+m}R.$$

If we make the replacements $\frac{x-x'}{R}=\cos\alpha$ and $\frac{y-y'}{R}=\sin\alpha$, then

$$\cos\alpha=\frac{x}{r},\qquad \sin\alpha=\frac{y}{r},\qquad x'/y'=x/y,\qquad u_x\approx -v_x\frac{m}{M},\qquad u_y\approx -v_y\frac{m}{M}.$$

The acceleration of the pulsar (2.23), at the initial moment with the initial conditions, can be simplified into the following form

$$\frac{d^2x}{dt^2}=-\left(1-\frac{v^2}{2c^2}\right)c^2S_{14}-\frac{v_x}{c^2}(a_xv_x+a_yv_y)+icv_yS_{12}, \tag{4.1a}$$

$$\frac{d^2y}{dt^2}=-\left(1-\frac{v^2}{2c^2}\right)c^2S_{24}-\frac{v_y}{c^2}(a_xv_x+a_yv_y)-icv_xS_{12}, \tag{4.1b}$$

where the components of the matrix $S=P(U,V)^T\phi P(U,V)$ should be calculated analogously to (3.1). In order to avoid large expressions for c^{-2} approximation, it is sufficient to use the components (3.1), replacing v_x by v_x-u_x and v_y by v_y-u_y. Hence for S_{14}, S_{24}, and S_{12} we obtain

$$S_{14}=\Bigg[-a_x\left(\frac{1}{\sqrt{1-\frac{(\mathbf{v}-\mathbf{u})^2}{c^2}}}-\frac{(v_x-u_x)^2}{2c^2}\right)$$
$$+a_y\frac{(v_x-u_x)(v_y-u_y)}{2c^2}-w_z(v_y-u_y)\Bigg]\frac{1}{c^2},$$

$$S_{24}=\Bigg[-a_y\left(\frac{1}{\sqrt{1-\frac{(\mathbf{v}-\mathbf{u})^2}{c^2}}}-\frac{(v_y-u_y)^2}{2c^2}\right)$$

$$+ a_x \frac{(v_x - u_x)(v_y - u_y)}{2c^2} + w_z(v_x - u_x)\bigg]\frac{1}{c^2},$$

$$S_{12} = -\frac{i}{c}\left[\frac{a_x}{c^2}(v_y - u_y) - \frac{a_y}{c^2}(v_x - u_x) + w_z\right].$$

According to (2.6) up to c^{-2}, we have

$$\frac{1 - \frac{u^2}{c^2}}{(1 - \frac{u^2}{c^2}\sin^2\theta)^{3/2}} = \frac{1 - \frac{u^2}{c^2}}{(1 - \frac{u^2}{c^2})^{3/2}(1 + \frac{u^2}{c^2}\cos^2\theta)^{3/2}}$$
$$= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2}\frac{m^2}{M^2}(\mathbf{r}' \cdot \frac{\mathbf{r}}{r})^2)^{3/2}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{1}{(1 + \frac{1}{c^2}\frac{m^2}{(M+m)^2}(\frac{dR}{dt})^2)^{3/2}}.$$

Using the results from Sects. 2.1 and 2.3, the components a_x, a_y, w_z are given by

$$a_x = -\frac{x}{r}\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{GM}{R^2}\left(1 - \frac{G(M + 2m)}{Rc^2}\right)\lambda^{-3},$$

$$a_y = -\frac{y}{r}\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{GM}{R^2}\left(1 - \frac{G(M + 2m)}{Rc^2}\right)\lambda^{-3},$$

$$w_z = \frac{Gm}{R^2c^2}\frac{v_x y - v_y x}{r}\lambda^{-3}, \quad \lambda = \sqrt{1 + \frac{1}{c^2}\frac{m^2}{(M + m)^2}\left(\frac{dR}{dt}\right)^2}.$$

Using the equalities between v_x, u_x; v_y, u_y; r, R and so on, (4.1) can be reduced to the following form

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{R}}{R}\frac{GM}{R^2}\left[1 + \frac{V^2}{c^2}\frac{M^2 + 4Mm + 2m^2}{(M + m)^2} - \frac{G(M + 2m)}{Rc^2}\right.$$
$$\left. - \frac{3}{2c^2}\frac{m^2}{(M + m)^2}\left(\frac{dR}{dt}\right)^2\right] + \mathbf{V}\frac{dR}{dt}\frac{GM}{R^2}\left(\frac{M}{M + m} + \frac{3}{2}\right)\frac{1}{c^2}, \tag{4.2}$$

where $\mathbf{V}$ is the relative velocity of the pulsar with respect to its companion.

Analogously to this acceleration, the acceleration of the body with mass M (pulsar companion) at the initial moment is given by

$$\frac{d^2\mathbf{r}'}{dt^2} = \frac{\mathbf{R}}{R}\frac{Gm}{R^2}\left[1 + \frac{V^2}{c^2}\frac{m^2 + 4Mm + 2M^2}{(M + m)^2} - \frac{G(2M + m)}{Rc^2}\right.$$
$$\left. - \frac{3}{2c^2}\frac{M^2}{(M + m)^2}\left(\frac{dR}{dt}\right)^2\right] - \mathbf{V}\frac{dR}{dt}\frac{Gm}{R^2}\left(\frac{m}{M + m} + \frac{3}{2}\right)\frac{1}{c^2}. \tag{4.3}$$

Subtracting (4.3) from (4.2), after some transformations we get

$$\frac{d^2\mathbf{R}}{dt^2} = -\frac{\mathbf{R}}{R}\frac{G(M + m)}{R^2}\left[1 + \frac{V^2}{c^2}\frac{M^2 + 5Mm + m^2}{(M + m)^2} - \frac{G(M^2 + 4Mm + m^2)}{Rc^2(M + m)}\right.$$
$$\left. - \frac{3}{2c^2}\frac{Mm}{(M + m)^2}\left(\frac{dR}{dt}\right)^2\right] + \mathbf{V}\frac{dR}{dt}\frac{G}{R^2}\frac{5M^2 + 6Mm + 5m^2}{2(M + m)c^2}. \tag{4.4}$$

All variables in (4.4) are related to the relative motion and so the assumption about the initial moment has no role. Now, having the system of (4.4) for the relative motion of a body with mass m with respect to the body with mass M, we can calculate the periastron shift in two steps, analogously as it has been made for the perihelion shift in Sect. 3. The first step consists of finding an equation analogous to (3.6) in the same way as in Sect. 3. In the second step we sum the first equation in (4.4) multiplied by $2V_x$ and the second equation of (4.4) multiplied by $2V_y$. That equation can be integrated and the value of V^2 can be found. After these two steps the periastron shift can be obtained. We present only the final results of these two steps avoiding the long algebraic and differential calculations.

The first step from the system (4.4) yields the following equation

$$\left(\frac{d\frac{1}{R}}{d\varphi}\right)^2 + \frac{1}{R^2} = V^2 C_2^{-2}\left[1 + \frac{5M^2 + 6Mm + 5m^2}{M+m}\frac{G}{Rc^2}\right]. \tag{4.5}$$

The second step is more complicated. Although $\frac{3}{2c^2}\frac{Mm}{(M+m)^2}(\frac{dR}{dt})^2$ has no influence in (4.5), it has a significant role in V^2, but we will see that it has no role in the periastron shift. In order to simplify the system (4.4), one can prove that $\frac{3}{2c^2}\frac{Mm}{(M+m)^2}(\frac{dR}{dt})^2$ has no influence on the periastron shift. The proof is standard and thus we omit it.

The second step from the modified system (4.4) yields

$$V^2 = 2\frac{G(M+m)}{R} - \frac{4G^2}{R^2c^2}(M^2+m^2) + \frac{C}{Rc^2} + K, \tag{4.6}$$

where C and K are mutually dependent constants, which have no role in the periastron shift. Now, analogously to (3.7) from (4.5) and (4.6), we get

$$\left(\frac{d\frac{1}{R}}{d\varphi}\right)^2 + \frac{1}{R^2} = A + B\frac{1}{R} + \frac{6G^2(M+m)^2}{C_2^2c^2}\frac{1}{R^2}.$$

Using that $C_2^2 = G(M+m)a_r(1-\epsilon^2)$, similar to the calculations for the perihelion shift in Sect. 3, for the periastron shift we obtain

$$\Delta\varphi = \frac{6\pi G(M+m)}{a_r(1-\epsilon^2)c^2}, \tag{4.7}$$

where a_r is the semi-major axis of the relative orbit and ϵ is the eccentricity of the orbit. This result is the same as in the GR.

5 Barycenter of Two Bodies

We proceed with the problem of two bodies considering their barycenter. We shall employ the same notations as in the previous section, and we will use the same coordinate system with the assumptions about the initial moment. Now, we prove that in the chosen coordinate system the barycenter coincides with the coordinate origin. According to (2.15) we have

$$\mathbf{r} = -\mathbf{r}'\frac{M(1+\frac{u^2}{2c^2} - \frac{Gm}{2Rc^2})}{m(1+\frac{v^2}{2c^2} - \frac{GM}{2Rc^2})} + \mathbf{r}_b\frac{M(1+\frac{u^2}{2c^2} - \frac{Gm}{2Rc^2}) + m(1+\frac{v^2}{2c^2} - \frac{GM}{2Rc^2})}{m(1+\frac{v^2}{2c^2} - \frac{GM}{2Rc^2})}.$$

Since

$$-\frac{M(1+\frac{u^2}{2c^2}-\frac{Gm}{2Rc^2})}{m(1+\frac{v^2}{2c^2}-\frac{GM}{2Rc^2})}=-\frac{M(1+\frac{V^2}{2c^2}\frac{m^2}{(M+m)^2}-\frac{Gm}{2Rc^2})}{m(1+\frac{V^2}{2c^2}\frac{M^2}{(M+m)^2}-\frac{GM}{2Rc^2})}$$
$$=-\frac{M}{m}\left(1-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{1}{2}\frac{G(M-m)}{Rc^2}\right),$$

at each moment it is satisfied

$$\frac{1}{M}\mathbf{r}=-\frac{1}{m}\mathbf{r}'\left(1-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)$$
$$+\mathbf{r}_b\left[\frac{1}{M}+\frac{1}{m}\left(1-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)\right]. \tag{5.1}$$

The accelerations (4.2) and (4.3) are given with respect to the coordinate system such that at the initial moment the coordinate center coincides with the barycenter $\mathbf{r}_b$ and $d\mathbf{r}_b/dt=(0,0,0)$ at the initial moment. But, if we replace $\mathbf{r}$ by $\mathbf{r}-\mathbf{r}_b$ in (4.2) and replace $\mathbf{r}'$ by $\mathbf{r}'-\mathbf{r}_b$ in (4.3), then (4.2) and (4.3) will be true for each t. Further, we will use these modified equations and will prove that at each moment

$$\frac{1}{M}\frac{d^2\mathbf{r}}{dt^2}=-\frac{1}{m}\frac{d^2}{dt^2}\left[\mathbf{r}'\left(1-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)\right]+\left(\frac{1}{M}+\frac{1}{m}\right)\frac{d^2\mathbf{r}_b}{dt^2},$$

i.e.

$$\frac{1}{M}\frac{d^2\mathbf{r}}{dt^2}+\frac{1}{m}\frac{d^2\mathbf{r}'}{dt^2}$$
$$=\frac{d^2}{dt^2}\left[\frac{\mathbf{R}}{M+m}\left(-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)\right]+\left(\frac{1}{M}+\frac{1}{m}\right)\frac{d^2\mathbf{r}_b}{dt^2}. \tag{5.2}$$

According to the modified equations (4.2) and (4.3), the left side of (5.2) is equal to

$$\frac{1}{M}\frac{d^2\mathbf{r}}{dt^2}+\frac{1}{m}\frac{d^2\mathbf{r}'}{dt^2}$$
$$=\frac{1}{M}\left[\frac{\mathbf{R}}{R}\frac{GM}{R^2}\frac{V^2}{c^2}\frac{M^2}{(M+m)^2}+\mathbf{V}\frac{dR}{dt}\frac{GM^2}{(M+m)R^2c^2}\right.$$
$$\left.+\frac{\mathbf{R}}{R}\frac{G^2Mm}{Rc^2}\frac{1}{R^2}+\frac{3}{2}\frac{\mathbf{R}}{R}\frac{GM}{R^2c^2}\frac{m^2}{(M+m)^2}\left(\frac{dR}{dt}\right)^2\right]$$
$$-\frac{1}{m}\left[\frac{\mathbf{R}}{R}\frac{Gm}{R^2}\frac{V^2}{c^2}\frac{m^2}{(M+m)^2}+\mathbf{V}\frac{dR}{dt}\frac{Gm^2}{(M+m)R^2c^2}\right.$$
$$\left.+\frac{\mathbf{R}}{R}\frac{G^2Mm}{Rc^2}\frac{1}{R^2}+\frac{3}{2}\frac{\mathbf{R}}{R}\frac{Gm}{R^2c^2}\frac{M^2}{(M+m)^2}\left(\frac{dR}{dt}\right)^2\right]+\left(\frac{1}{M}+\frac{1}{m}\right)\frac{d^2\mathbf{r}_b}{dt^2}$$
$$=\frac{\mathbf{R}}{R}\frac{G}{R^2}\frac{V^2}{c^2}\frac{M-m}{M+m}+\mathbf{V}\frac{dR}{dt}\frac{G}{R^2c^2}\frac{M-m}{M+m}$$

$$-\frac{\mathbf{R}}{R}\frac{G^2(M-m)}{Rc^2}\frac{1}{R^2}-\frac{3}{2}\frac{\mathbf{R}}{R}\frac{G}{R^2c^2}\frac{M-m}{M+m}\left(\frac{dR}{dt}\right)^2+\left(\frac{1}{M}+\frac{1}{m}\right)\frac{d^2\mathbf{r}_b}{dt^2}.$$

Hence, it is sufficient to prove that

$$\begin{aligned}&\frac{d^2}{dt^2}\left[\frac{\mathbf{R}}{M+m}\left(-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)\right]\\&\quad=\frac{\mathbf{R}}{R}\frac{G}{R^2}\frac{V^2}{c^2}\frac{M-m}{M+m}+\mathbf{V}\frac{dR}{dt}\frac{G}{R^2c^2}\frac{M-m}{M+m}\\&\qquad-\frac{\mathbf{R}}{R}\frac{G^2(M-m)}{Rc^2}\frac{1}{R^2}-\frac{3}{2}\frac{\mathbf{R}}{R}\frac{G}{R^2c^2}\frac{M-m}{M+m}\left(\frac{dR}{dt}\right)^2.\end{aligned}$$

After multiplication with $\frac{2c^2}{G}\frac{M+m}{M-m}$, we should prove that

$$\begin{aligned}&\frac{d^2}{dt^2}\frac{\mathbf{R}}{R}-\frac{1}{G(M+m)}\frac{d^2}{dt^2}(\mathbf{R}(\mathbf{V}\cdot\mathbf{V}))\\&\quad=2\frac{\mathbf{R}}{R}\frac{1}{R^2}V^2+2\mathbf{V}\frac{dR}{dt}\frac{1}{R^2}-2\frac{\mathbf{R}}{R}\frac{(M+m)G}{R^3}-3\frac{\mathbf{R}}{R^3}\left(\frac{dR}{dt}\right)^2.\end{aligned}\tag{5.3}$$

Using the identities

$$\frac{d^2}{dt^2}\frac{\mathbf{R}}{R}=-2\mathbf{V}\frac{dR}{dt}\frac{1}{R^2}-\frac{\mathbf{R}}{R^3}V^2+3\frac{\mathbf{R}}{R^3}\left(\frac{dR}{dt}\right)^2\quad\text{and}\quad\frac{dR}{dt}=\mathbf{V}\cdot\frac{\mathbf{R}}{R},$$

the identity (5.3) is equivalent to

$$\begin{aligned}&-\frac{1}{G(M+m)}\frac{d^2}{dt^2}(\mathbf{R}(\mathbf{V}\cdot\mathbf{V}))\\&\quad=3\frac{\mathbf{R}}{R^3}V^2+4\mathbf{V}\frac{dR}{dt}\frac{1}{R^2}-6\frac{\mathbf{R}}{R^3}\left(\frac{dR}{dt}\right)^2-2\frac{\mathbf{R}}{R}\frac{(M+m)G}{R^3}.\end{aligned}\tag{5.4}$$

A straight calculation and using the identity $V^2=\frac{2G(M+m)}{R}+\text{const.}$ one verifies the identity (5.4), i.e. (5.2).

From (5.1) and (5.2) it follows that

$$\left(\frac{1}{M}+\frac{1}{m}\right)\frac{d^2\mathbf{r}_b}{dt^2}=\frac{d^2}{dt^2}\left[\mathbf{r}_b\left(\frac{1}{M}+\frac{1}{m}\left(1-\frac{M-m}{2(M+m)}\frac{V^2}{c^2}+\frac{G(M-m)}{2Rc^2}\right)\right)\right],$$

i.e.

$$\frac{d^2}{dt^2}\left[\mathbf{r}_b\left(\frac{G(M+m)}{R}-V^2\right)\right]=0.\tag{5.5}$$

Hence it follows

$$\mathbf{r}_b\left(\frac{G(M+m)}{R}-V^2\right)=A+Bt,$$

where A and B are constants, assuming that the initial moment is $t=0$. Since $\mathbf{r}_b=(0,0,0)$ for $t=0$ by assumption, we obtain $A=0$. By assumption we also have $d\mathbf{r}_b/dt=(0,0,0)$

for $t = 0$, and hence $B = 0$. Thus, $\mathbf{r}_b \equiv (0,0,0)$, if we assume that $\mathbf{r}_b = (0,0,0)$ and $d\mathbf{r}_b/dt = (0,0,0)$ at the initial moment. As a consequence, the formulae (4.2) and (4.3) are true not only at the initial moment, but along the whole trajectory.

6 Equations of Motion for *n*-Body Problem and Their Relationship with the GR Equations

Now, we will consider the n-body problem, i.e. the equations of motion of n bodies with arbitrary masses. Assume that all bodies are compressed into points, and hence we neglect their angular momenta. We will obtain the equations of motion in explicit form using 3-vectors of distances between the bodies, and their velocities and accelerations.

Let a system of n bodies with masses $m_1, m_2, \dots, m_n$, with initial positions and initial velocities be given. We denote by $\mathbf{r}_k$ and $\mathbf{v}_k$ the 3-radius vector and 3-vector of velocity of the body with mass m_k, and denote $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ for $i \neq j$. We will write the equations only for the motion of the j-th body under the gravitation of the i-th body and then follows summation for all $i \neq j$.

Analogously as in Sect. 4, the components of the matrix S are

$$S_{14} = \left[-a_x\left(1 + \frac{(v_y - u_y)^2}{2c^2} + \frac{(v_z - u_z)^2}{2c^2}\right) + a_y\frac{(v_x - u_x)(v_y - u_y)}{2c^2}\right.$$
$$\left. + a_z\frac{(v_x - u_x)(v_z - u_z)}{2c^2} + w_y(v_z - u_z) - w_z(v_y - u_y)\right]\frac{1}{c^2},$$
$$S_{24} = \left[a_x\frac{(v_x - u_x)(v_y - u_y)}{2c^2} - a_y\left(1 + \frac{(v_x - u_x)^2}{2c^2} + \frac{(v_z - u_z)^2}{2c^2}\right)\right.$$
$$\left. + a_z\frac{(v_y - u_y)(v_z - u_z)}{2c^2} + w_z(v_x - u_x) - w_x(v_z - u_z)\right]\frac{1}{c^2},$$
$$S_{34} = \left[a_x\frac{(v_x - u_x)(v_z - u_z)}{2c^2} + a_y\frac{(v_y - u_y)(v_z - u_z)}{2c^2}\right.$$
$$\left. - a_z\left(1 + \frac{(v_x - u_x)^2}{2c^2} + \frac{(v_y - u_y)^2}{2c^2}\right) + w_x(v_y - u_y) - w_y(v_x - u_x)\right]\frac{1}{c^2},$$
$$S_{12} = -\frac{i}{c}\left[\frac{a_x}{c^2}(v_y - u_y) - \frac{a_y}{c^2}(v_x - u_x) + w_z\right],$$
$$S_{23} = -\frac{i}{c}\left[\frac{a_y}{c^2}(v_z - u_z) - \frac{a_z}{c^2}(v_y - u_y) + w_x\right],$$
$$S_{31} = -\frac{i}{c}\left[\frac{a_z}{c^2}(v_x - u_x) - \frac{a_x}{c^2}(v_z - u_z) + w_y\right],$$

where $(v_x, v_y, v_z) = \mathbf{v}_j$ and $(u_x, u_y, u_z) = \mathbf{v}_i$. Notice that the orthogonal matrix P from (2.18) besides the Lorentz transformation of velocity $\mathbf{u} - \mathbf{v}$ contains also a space rotation determined by the 3-vector $(\mathbf{u} \times \mathbf{v})/c^2$. This small space rotation will be taken into account by changing the components of the tensor ϕ, i.e. of a_x, a_y, a_z, while the influence on w_x, w_y, w_z is of order c^{-4}. The previous equations can be written in the following compact form

$$\mathbf{S}=\left[\mathbf{a}\left(1+\frac{(\mathbf{v}_j-\mathbf{v}_i)^2}{2c^2}\right)-\frac{\mathbf{v}_j-\mathbf{v}_i}{2c^2}(\mathbf{a}\cdot(\mathbf{v}_j-\mathbf{v}_i))+(\mathbf{v}_j-\mathbf{v}_i)\times\mathbf{w}\right]\frac{1}{c^2},$$
$$\mathbf{S}^*=-\frac{i}{c^3}[\mathbf{a}\times(\mathbf{v}_j-\mathbf{v}_i)]-\frac{i}{c}\mathbf{w}, \tag{6.1}$$

where $\mathbf{S}=(S_{41},S_{42},S_{43})$ and $\mathbf{S}^*=(S_{23},S_{31},S_{12})$.

Further, the components of the tensor ϕ caused by the body with mass m_i are given by

$$a_x=-(x-x')\frac{Gm_i}{R^3\lambda^3\mu}\left(1+\frac{u^2}{2c^2}\right)-\frac{Gm_i}{r_{ij}^3c^2}\Big[(\mathbf{r}_j-\mathbf{r}_i)\times((\mathbf{r}_j-\mathbf{r}_i)\times\dot{\mathbf{v}}_i)\Big]_x$$
$$-(y-y')\frac{v_xu_y-v_yu_x}{c^2}\frac{Gm_i}{R^3}+(z-z')\frac{v_zu_x-v_xu_z}{c^2}\frac{Gm_i}{R^3},$$
$$a_y=-(y-y')\frac{Gm_i}{R^3\lambda^3\mu}\left(1+\frac{u^2}{2c^2}\right)-\frac{Gm_i}{r_{ij}^3c^2}\Big[(\mathbf{r}_j-\mathbf{r}_i)\times((\mathbf{r}_j-\mathbf{r}_i)\times\dot{\mathbf{v}}_i)\Big]_y$$
$$-(z-z')\frac{v_yu_z-v_zu_y}{c^2}\frac{Gm_i}{R^3}+(x-x')\frac{v_xu_y-v_yu_x}{c^2}\frac{Gm_i}{R^3},$$
$$a_z=-(z-z')\frac{Gm_i}{R^3\lambda^3\mu}\left(1+\frac{u^2}{2c^2}\right)-\frac{Gm_i}{r_{ij}^3c^2}\Big[(\mathbf{r}_j-\mathbf{r}_i)\times((\mathbf{r}_j-\mathbf{r}_i)\times\dot{\mathbf{v}}_i)\Big]_z$$
$$-(x-x')\frac{v_zu_x-v_xu_z}{c^2}\frac{Gm_i}{R^3}+(y-y')\frac{v_yu_z-v_zu_y}{c^2}\frac{Gm_i}{R^3},$$
$$w_x=\frac{Gm_i}{c^2R^3}[(y-y')u_z-u_y(z-z')],$$
$$w_y=\frac{Gm_i}{c^2R^3}[(z-z')u_x-u_z(x-x')],\qquad w_z=\frac{Gm_i}{c^2R^3}[(x-x')u_y-u_x(y-y')],$$

where $(x,y,z)=\mathbf{r}_j$, $(x',y',z')=\mathbf{r}_i$, $R=r_{ij}$, $\lambda=1+\frac{1}{2c^2}(\mathbf{u}\cdot\frac{\mathbf{r}_j-\mathbf{r}_i}{r_{ij}})^2$,

$$\mu=\left(1+\frac{Gm_i}{r_{ij}c^2}\right)\left(1+\frac{2Gm_j}{r_{ij}c^2}\right)\prod_{k\neq i,j}\left[\left(1+\frac{Gm_k}{r_{ki}c^2}\right)\left(1+\frac{Gm_k}{r_{kj}c^2}\right)\right].$$

The last two terms in a_x, a_y, and a_z are added as influence from the space rotation given by the vector $(\mathbf{u}\times\mathbf{v})/c^2$. These equalities in vector form can be written as

$$\mathbf{a}=-\frac{(\mathbf{r}_j-\mathbf{r}_i)Gm_i}{r_{ij}^3}\Bigg[1-\frac{3}{2}\frac{[\mathbf{v}_i\cdot(\mathbf{r}_j-\mathbf{r}_i)]^2}{r_{ij}^2c^2}$$
$$-\frac{G(m_i+2m_j)}{r_{ij}c^2}-\sum_{k\neq i,j}\left(\frac{Gm_k}{r_{ki}c^2}+\frac{Gm_k}{r_{kj}c^2}\right)+\frac{v_i^2}{2c^2}\Bigg]$$
$$-\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{r}_j-\mathbf{r}_i)\times((\mathbf{r}_j-\mathbf{r}_i)\times\dot{\mathbf{v}}_i)-\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{r}_j-\mathbf{r}_i)\times(\mathbf{v}_j\times\mathbf{v}_i). \tag{6.2}$$

For the equations of motion of the j-th body, analogously to (4.1) we obtain

$$\frac{d^2x}{dt^2}=-\left(1-\frac{v^2}{2c^2}\right)(S_{41}v_x^2+S_{42}v_xv_y+S_{43}v_xv_z)$$
$$+ic\left(1-\frac{v^2}{2c^2}\right)(S_{12}v_y+S_{13}v_z+icS_{14}),$$
$$\frac{d^2y}{dt^2}=-\left(1-\frac{v^2}{2c^2}\right)(S_{41}v_xv_y+S_{42}v_y^2+S_{43}v_yv_z)$$
$$+ic\left(1-\frac{v^2}{2c^2}\right)(S_{21}v_x+S_{23}v_z+icS_{24}),$$
$$\frac{d^2z}{dt^2}=-\left(1-\frac{v^2}{2c^2}\right)(S_{41}v_xv_z+S_{42}v_yv_z+S_{43}v_z^2)$$
$$+ic\left(1-\frac{v^2}{2c^2}\right)(S_{31}v_x+S_{32}v_y+icS_{34}).$$

In vector form they can be written as

$$\frac{d^2\mathbf{r}_j}{dt^2}=-\mathbf{v}_j(\mathbf{S}\cdot\mathbf{v})\left(1-\frac{v^2}{2c^2}\right)+\mathbf{S}\left(1-\frac{v^2}{2c^2}\right)c^2+ic\left(1-\frac{v^2}{2c^2}\right)(\mathbf{v}\times\mathbf{S}^*). \tag{6.3}$$

Finally, after summation for all $i\neq j$ and after many transformations, (6.3) become

$$\frac{d^2\mathbf{r}_j}{dt^2}=\sum_{i\neq j}\Bigg\{-\frac{(\mathbf{r}_j-\mathbf{r}_i)Gm_i}{r_{ij}^3}\Bigg[1-\frac{3}{2}\frac{[\mathbf{v}_i\cdot(\mathbf{r}_j-\mathbf{r}_i)]^2}{r_{ij}^2c^2}$$
$$-\frac{G(m_i+2m_j)}{r_{ij}c^2}-\sum_{k\neq i,j}\left(\frac{Gm_k}{r_{ki}c^2}+\frac{Gm_k}{r_{kj}c^2}\right)+\frac{v_i^2}{c^2}-2\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{c^2}+\frac{(\mathbf{v}_i-\mathbf{v}_j)^2}{c^2}\Bigg]$$
$$-\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{r}_j-\mathbf{r}_i)\times((\mathbf{r}_j-\mathbf{r}_i)\times\dot{\mathbf{v}}_i)+\frac{3}{2}\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{v}_j-\mathbf{v}_i)[(\mathbf{r}_j-\mathbf{r}_i)\cdot(\mathbf{v}_j-\mathbf{v}_i)]$$
$$+\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{v}_j-\mathbf{v}_i)[(\mathbf{r}_j-\mathbf{r}_i)\cdot\mathbf{v}_j]\Bigg\}. \tag{6.4}$$

Now, let us distinguish the terms in (6.4) which are Lorentz invariant. From (6.4) we obtain

$$\frac{d^2\mathbf{r}_j}{dt^2}=\sum_{i\neq j}\Bigg\{-\frac{(\mathbf{r}_j-\mathbf{r}_i)Gm_i}{r_{ij}^3}\left[1-\frac{3}{2}\frac{[\mathbf{v}_i\cdot(\mathbf{r}_j-\mathbf{r}_i)]^2}{r_{ij}^2c^2}+\frac{v_i^2}{c^2}-2\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{c^2}\right]$$
$$+\frac{Gm_i}{r_{ij}^3c^2}(\mathbf{v}_j-\mathbf{v}_i)[(\mathbf{r}_j-\mathbf{r}_i)\cdot\mathbf{v}_j]\Bigg\}+\text{Lorentz invariant terms}. \tag{6.5}$$

Notice that the Einstein-Infeld-Hoffmann (EIH) equations [15] can also be written in the same form (6.5), and hence the conclusion that (6.4) *differ from the Einstein-Infeld-Hoffmann equations by Lorentz invariant terms of order* c^{-2}. The different nature of the coordinate systems for (6.4) and (6.5) does not permit them to be identical.

Now let us consider a special case of two bodies. From (6.4) we can determine the relative orbit and then we can compare it with the corresponding GR relative orbit via EIH equations. It is interesting that *assuming the change of time* $d\bar{t} = (1 + \frac{3}{2}\frac{G(M+m)}{Rc^2})dt$, *the relative orbit from* (6.4) *maps into the relative orbit according to the GR*. Since the reparameterization of the trajectories by the time does not change the "space trajectories", now we generalize the results about the periastron shift, because now the coordinate system may not rest at the barycenter of the two bodies and they may not move in a single plane.

7 Some Remarks and Conclusions

Using orthonormal frames enables a deep study on the effects where angles and precessions are measured and the results regarding the barycenter of two bodies are the same as in GR. However, not everything was considered and we complete it now.

7.1 The Analog of the PPN Parameter γ

In the GR, the non-zero PPN parameters γ and β are both equal to 1. So, it is natural to ask what is their meaning in this approach. The formulae for the deflection of the light rays, geodetic precession and the frame dragging effect lead to the following conclusion: The same formulae which are obtained for $\gamma = 1$ in GR, here are obtained via the equations of motion (2.23). But, if we omit the matrix P (and P^T also) in (2.23), then we obtain formulae which are identical for $\gamma = 0$ in the PPN approach, for example in deflection of the light rays near the Sun, geodetic precession and the frame dragging. Hence, *the appearance of the matrix P in* (2.23) *corresponds to* $\gamma = 1$. The previous statement about $\gamma = 0$ and $\gamma = 1$ can be verified directly from the equations of motion. The Lorentz force shows that for the acceleration of a charged particle one should know only the tensor of electromagnetic field, which is analogous to ϕ_{ij}, and it is not necessary to know the velocity of the source and the tensor P has no role there. Thus, we can intuitively say that $\gamma = 0$ in the electrodynamics. Now we clearly see the similarity and the differences between the electrodynamics and gravitation. Notice that the Larmour's theorem suggests connection between magnetic field and the angular velocity, hence simultaneously with the electromagnetic tensor it is natural to introduce and to consider the tensor ϕ.

The Coriolis force $\mathbf{f} = 2m\mathbf{v} \times \mathbf{w}$ is obtained in (3.2) where the tensor P has the essential role. By omitting the matrix P in (2.23), the result would be $\mathbf{f} = m\mathbf{v} \times \mathbf{w}$. Thus, we can write $\mathbf{f} = (1 + \gamma)m\mathbf{v} \times \mathbf{w}$. The previous discussion and also the Larmour's theorem are the reason why in many formulae comparing the angular velocity with the magnetic field, the coefficient 2 appears.

7.2 Gravitational Radiation

Further, let us discuss the gravitational radiation. The intensity of the quadrupole electromagnetic radiation is given by [19]

$$I = \frac{1}{180c^5}\left(\frac{d^3 D_{\alpha\beta}}{dt^3}\right)^2. \tag{7.1}$$

Having in mind that the intensity of the electromagnetic radiation is proportional to H^2 and, $H \sim \frac{1}{2}w$, we see that in case of gravitation the corresponding intensity should be $2^2 = 4$

times larger (assuming a system of units where $G = 1$), i.e., it should be

$$I = \frac{G}{45c^5}\left(\frac{d^3 D_{\alpha\beta}}{dt^3}\right)^2. \tag{7.2}$$

This formula (7.2) is well known for the gravitational radiation [19] according to the GR. Moreover, it is known [19] that if the charges of the particles in one system are proportional to the corresponding masses of the particles, then there will not exist a dipole electromagnetic radiation. In case of gravitation, this means that there will not exist dipole gravitational radiation in case of two bodies.

7.3 The Analog of the PPN Parameter β

Analogously to the PPN parameter β, in flat Minkowski space we determine a parameter β^* via the expansion of the coefficient μ:

$$\mu = 1 + \frac{GM}{rc^2} + \beta^*\left(\frac{GM}{rc^2}\right)^2 + \cdots . \tag{7.3}$$

Now one can calculate that the perihelion shift is given by

$$\Delta\varphi = (6\gamma + 2\beta^*)\frac{GM\pi}{ac^2(1-\epsilon^2)}, \tag{7.4}$$

and since $\beta^* = 0$, the total perihelion shift is a consequence of appearance of the tensor P. Comparing this formula with the corresponding PPN formula, we see that the PPN parameter β corresponds to $2 - \gamma + \beta^*$, which shows that the tensor P has influence not only on g_{11}, g_{22}, and g_{33}, but also to g_{44}.

7.4 Lagrangian

In Sect. 3 the energy of a particle which moves in a gravitational field was derived, and it was given by the Lorentz invariant form (3.4d). Assume that we have a source of gravitation at the coordinate center, which rests with respect to the chosen coordinate system. Then the Lagrangian is given by

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + mc^2\ln\left(1 + \frac{GM}{rc^2}\right).$$

Indeed, a direct calculation shows that the Hamiltonian function is given by

$$\mathcal{H} = \mathbf{v}\frac{\partial L}{\partial \mathbf{v}} - L = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2\ln\left(1 + \frac{GM}{rc^2}\right)$$

and it is a constant according to (3.4c). Further the Euler-Lagrange equations can be written in the following equations

$$\frac{dV_i}{ds} = \phi_{ij}V_j, \tag{7.5}$$

for $i = 1, 2, 3$. Compared with (2.23) we notice that if we dismiss the tensor P and also P^T, we obtain (7.5). On the other side, we mentioned in Sect. 3, that (7.5) lead to the same

Hamiltonian function. Indeed, the appearance of the tensor P means presence of a force, which does not do action, i.e. preserves the energy of the test body in the gravitational field. Finally, notice that analogously to (3.4d), the Lagrangian can be written in the following Lorentz invariant form

$$L = -\frac{mc^2}{U_i V_i} + mc^2 \ln\left(1 + \frac{GM}{rc^2}\right), \tag{7.6}$$

where U_i is the 4-vector of velocity of the gravitational body, V_i is the 4-vector of velocity of the test body with negligible mass m and the distance r is determined in the system where the gravitational body rests.

7.5 The Field Equations

According to the discussion in Sect. 2.2, the gravitational potential in case of many distinct bodes with point masses is given by

$$\mu = \prod_i \left(1 + \frac{Gm_i}{r_i c^2}\right) \tag{7.7}$$

where the distance r_i between the considered test body and the i-th gravitational body is determined in the system where the i-th gravitational body rests. The scalar μ can be interpreted as a scalar which is related to the gravitational redshift caused by many bodies.

In case of a mass distribution given by the mater density ρ, the potential μ is given by

$$\ln \mu = \int_{V'} \ln\left(1 + \frac{G\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|c^2} dV'\right), \tag{7.8}$$

where the distance $|\mathbf{r} - \mathbf{r}'|$ is defined analogous as in (7.7). Notice that it is wrong if we write (7.8) in the form

$$\ln \mu = \int_{V'} \frac{G\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|c^2} dV', \tag{7.8$'$}$$

because ρ is closer to Dirac function, where $\int \rho dV' = M$, than to a bound function. Indeed, assuming that ρ is bound function and then (7.8′) is true, using that $\int \rho dV' = M$ in case of one body, we would obtain $\ln \mu = \frac{GM}{rc^2}$ and hence $\mu = \exp(\frac{GM}{rc^2})$, which is a wrong result.

Now the density ρ satisfies the following partial differential equations

$$\frac{\partial \phi_{ij}}{\partial x_k} + \frac{\partial \phi_{jk}}{\partial x_i} + \frac{\partial \phi_{ki}}{\partial x_j} = 0 \tag{7.9}$$

and

$$\frac{\partial \phi_{ij}}{\partial x_j} = \frac{4\pi G\rho}{c^2} U_i \tag{7.10}$$

where U_i is the field of 4-vector of velocity of the mater distribution. They are the field equations and they are completely analogous to the Maxwell's equations for electrodynamics. If we put $i = 4$ and $u \approx 0$ in (7.10), we just obtain the Poisson's equation.

7.6 Short Discussion About the Einstein-Infeld-Hoffmann Equations

At the end we try to explain why EIH equations correspond to the case 4 in Sect. 1. There are two main reasons: (i) While there is no privileged metric in any metric gravitational theory for determining any (invariant) scalar, for example curvature scalar, for calculating some noninvariant scalars we really need a privileged system. For example, we apply the equations of motion from any metric theory to find many scalars (like perihelion shift per orbit) assuming a priori that the corresponding equations are related to a flat manifold, but not curved. These calculations may lead to satisfactory results, which really happens, only in an inertial system, i.e. far from the gravitational fields. (ii) The fact that (6.4) differ from the EIH equations for Lorentz invariant terms also suggests that both equations are given with respect to an observer far from gravitation. The equations of motion according to an observer inside the gravitational field are much more complicated.

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